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# Characterization of all the supersymmetric solutions of gauged $N = 1, D = 5$ supergravity

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Abstract: We find a complete characterization of all the supersymmetric solutions of non-Abelian gauged $N = 1, d = 5$ supergravity coupled to vector multiplets and hypermultiplets: the generic forms of the metrics as functions of the scalars and vector fields plus the equations that all these must satisfy. These equations are now a complicated non-linear system and there it seems impossible to produce an algorithm to construct systematically all supersymmetric solutions.

Keywords: Extended Supersymmetry, Superstring Vacua.

## Contents



## 1. Introduction

Supersymmetric classical solutions of supergravity theories have played and continue playing a crucial role in many important developments such as $AdS/CFT$ correspondence, stringy black-hole Physics etc. This is why a great effort has been made to find , classify, or, at least, characterize all of them.

This program has been carried out to completion in several lower-dimensional theories. The first work of this kind was carried in pure, ungauged, $N = 2, d = 4$ supergravity by Tod in his pioneering 1983 work [1] and it has been extended to the gauged case in ref. [2] and to include the coupling to general (ungauged) vector multiplets and hypermultiplets in refs. [3] and [4], respectively. Pure $N = 4, d = 4$ supergravity was dealt with in refs. [5, 6]. The minimal $N = 1, d = 5$ theory was worked out in ref. [7] and the results were extended to the gauged case in ref. [8], and the coupling to an arbitrary number of vector multiplets and their Abelian gaugings was considered in refs. [9, 10].[1] The inclusion of (ungauged)

[1]Previous work on these theories can be found in refs. [11, 12].

hypermultiplets was considered in [13][2] and the goal of this paper is to further extend these results to include non-Abelian gaugings.

The minimal $d = 6$ SUGRA was dealt with in refs. [17, 18], some gaugings were considered in ref. [19] and the coupling to hypermultiplets has been fully solved in ref. [20].

All the works cited are essentially based on the method pioneered by Tod and generalized by Gauntlett *et al.* in ref. [7].[3] This method consists on assuming the existence of one Killing spinor and then deriving consistency conditions for this to be true. These conditions can be conveniently computed on tensors constructed as bilinears of the Killing spinors and constrain the form of the fields of the supersymmetric configuration. Finally the equations of motion have to be imposed on the constrained configurations, leading to simpler equations involving the undetermined components of the fields. This is the method that we are going to use here.

In the simplest cases (ungauged supergravities coupled to vector multiplets) the equations that have to be solved are uncoupled, typically linear, and can be solved in a systematic way. We can then construct supersymmetric solutions for those theories in a systematic way. The coupling to hypermultiplets [4, 13, 20] introduces new equations which, not only are non-linear but are coupled and have to be solved simultaneously. In particular one finds supersymmetry implies that the hyperscalar functions have to solve a nonlinear equation and, at the same time, they must be such that the pullback of the quaternionic SU(2) connection is gauge equivalent to the anti-selfdual part of the spin connection of the base space. Finding base spaces and hyperscalars that satisfy these two conditions is highly non-trivial and it is not known how to do it systematically. Still, once those two conditions are solved, the remaining equations are linear and uncoupled (Laplace equations for independent functions on the base space).

As we are going to see, the introduction of non-Abelian gaugings leads to yet more non-linear and coupled equations. This was to be expected since, for instance, the requirement of having unbroken supersymmetry in Euclidean $d = 4$ super-Yang-Mills theories still leaves us with non-linear equations to be solved, namely finding gauge potentials that give self- or anti-self-dual field strengths. In the case that we are going to study, timelike supersymmetry implies that the hyperscalar functions have to solve a nonlinear equation which involves, not only the hyperscalars, but the gauge potentials and the scalars belonging to the vector multiplets which, at the same time, must satisfy other equations. Simultaneously, the hyperscalar functions must be such that the covariant pullback of the quaternionic SU(2) connection is gauge equivalent to the anti-selfdual part of the spin connection of the base space. This is another condition that involves the hyperscalars, the gauge connection and the base space metric.

Our results are, thus, less satisfactory than in the simplest cases, even if they are complete characterizations of the necessary and sufficient conditions for any field configuration to be a supersymmetric solution. Constructing supersymmetric solutions of these theories is a difficult problem even though we know the minimal set of equations that should be

[2]Previous partial results on that problem were presented in refs. [14 – 16].

[3]Further works based on the alternative methods of spinorial geometry are refs. [21, 22].



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solved.[4] We, thus, leave for future work the construction of particular examples [23].

This paper is organized as follows: in section 2 we present the fields, Lagrangian and supersymmetry transformation rules of the theories that we are going to study. In section 3 we study the necessary and sufficient conditions for a configuration to be supersymmetric. As usual, we study separately the case in which the Killing vector constructed as bilinear of the Killing spinor is timelike (section 3.1) and null (section 3.2). In section 4 we present our conclusions. The appendix contains some useful formulae used in the main text concerning the gauging of isometries and the definition and meaning of the momentum map.

## 2. $N = 1, D = 5$ supergravity with gaugings

In this section we are going to briefly describe the action, equations of motion and supersymmetry transformation rules of gauged $N = 1, d = 5$ supergravities,[5] which we take from ref. [30], relying in the description of the ungauged theories given in ref. [13], whose conventions we follow. Appendix A contains a description of the gauging of the isometries of the scalar manifolds of the theory in which the definitions of the covariant derivatives $\mathfrak{D}$, gauge transformations and momentum map $\vec{P}_I$ can be found.

The bosonic action of $N = 1, d = 5$ gauged supergravity is given by

$$\begin{aligned} S = \int d^5x\sqrt{g}\,\Big\{ & R + \frac{1}{2}g_{xy}\mathfrak{D}_\mu\phi^x\mathfrak{D}^\mu\phi^y + \frac{1}{2}g_{XY}\mathfrak{D}_\mu q^X\mathfrak{D}^\mu q^Y + \mathcal{V}(\phi,q) - \frac{1}{4}a_{IJ}F^{I\,\mu\nu}F^J{}_{\mu\nu} \\ & + \frac{1}{12\sqrt{3}}C_{IJK}\frac{\varepsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{g}}\left(F^I{}_{\mu\nu}F^J{}_{\rho\sigma}A^K{}_\alpha - \frac{1}{2}gf_{LM}{}^I F^J{}_{\mu\nu}A^K{}_\rho A^L{}_\sigma A^M{}_\alpha \right. \\ & \left. + \frac{1}{10}g^2 f_{LM}{}^I f_{NP}{}^J A^K{}_\mu A^L{}_\nu A^M{}_\rho A^N{}_\sigma A^P{}_\alpha\right)\Big\}\,, \end{aligned} \tag{2.1}$$

where

$$\mathcal{V}(\phi,q) = g^2\left(4C_{IJK}h^I\vec{P}^J\cdot\vec{P}^K - \frac{3}{2}h^I h^J k_I{}^X k_J{}^Y g_{XY}\right)\,, \tag{2.2}$$

is the potential for the scalars. In the limit of pure supergravity, $n_H = n_V = 0$, $\mathcal{V}$ becomes a cosmological constant.

The equations of motion, for which we use the same notation as in ref. [13], are

$$\begin{aligned} \mathcal{E}_{\mu\nu} = G_{\mu\nu} & - \frac{1}{2}a_{IJ}\left(F^I{}_\mu{}^\rho F^J{}_{\nu\rho} - \frac{1}{4}g_{\mu\nu}F^{I\,\rho\sigma}F^J{}_{\rho\sigma}\right) + \frac{1}{2}g_{xy}\left(\mathfrak{D}_\mu\phi^x\mathfrak{D}_\nu\phi^y - \frac{1}{2}g_{\mu\nu}\mathfrak{D}_\rho\phi^x\mathfrak{D}^\rho\phi^y\right) \\ & + \frac{1}{2}g_{XY}\left(\mathfrak{D}_\mu q^X\mathfrak{D}_\nu q^Y - \frac{1}{2}g_{\mu\nu}\mathfrak{D}_\rho q^X\mathfrak{D}^\rho q^Y\right) - \frac{1}{2}g_{\mu\nu}\mathcal{V}\,, \end{aligned} \tag{2.3}$$

[4] A solutions could be immediately constructed, though, by dimensionally reducing the 6-dimensional dyonic string of ref. [20].

[5] Gauging of $N = 1, d = 5$ supergravity theories was first considered in ref. [24, 25]. More general gaugings in the vector multiplet sector plus the tensor multiplet sector (which we are not considering here) were considered in refs. [26, 27] and hypermultiplets and their gaugings were considered in ref. [28]. More general gaugings of the tensor multiplets and $N = 1, d = 5$ supergravities that do not admit actions were considered in ref. [29].



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$$g^{xy}\mathcal{E}_y = \mathfrak{D}_\mu \mathfrak{D}^\mu \phi^x + \frac{1}{4}\partial^x a_{IJ} F^{I\,\rho\sigma} F^J{}_{\rho\sigma} - \partial^x \mathcal{V} \tag{2.4}$$

$$g^{XY}\mathcal{E}_Y = \mathfrak{D}_\mu \mathfrak{D}^\mu q^X - \partial^X \mathcal{V}\,, \tag{2.5}$$

$$\mathcal{E}_I{}^\mu = \mathfrak{D}_\nu F_I{}^{\nu\mu} + \frac{1}{4\sqrt{3}}\frac{\varepsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{g}} C_{IJK} F^J{}_{\nu\rho} F^K{}_{\sigma\alpha} + g\left(k_{I\,x}\mathfrak{D}^\mu \phi^x + k_{I\,X}\mathfrak{D}^\mu q^X\right)\,. \tag{2.6}$$

The supersymmetry transformation rules for the fermionic fields, evaluated on vanishing fermions, are

$$\delta_\epsilon \psi^i_\mu = \mathfrak{D}_\mu \epsilon^i - \frac{1}{8\sqrt{3}} h_I F^{I\,\alpha\beta}\left(\gamma_{\mu\alpha\beta} - 4 g_{\mu\alpha}\gamma_\beta\right)\epsilon^i + \frac{1}{2\sqrt{3}} g h^I \gamma_\mu \epsilon^j P_{I\,j}{}^i\,, \tag{2.7}$$

$$\delta_\epsilon \lambda^{ix} = \frac{1}{2}\left(\not{\mathfrak{D}}\phi^x - \frac{1}{2}h^x_I \not{F}^I\right)\epsilon^i + g h^x_I \epsilon^j P^I{}_j{}^i\,, \tag{2.8}$$

$$\delta_\epsilon \zeta^A = \frac{1}{2} f_X{}^{iA}\left(\not{\mathfrak{D}} q^X + \sqrt{3} g h^I k_I{}^X\right)\epsilon_i\,. \tag{2.9}$$

The supersymmetry transformation rules of the bosonic fields are exactly the same as in the ungauged case [13]. This implies that the form of the Killing spinor identities (KSIs) relating the bosonic equations of motion that one can derive from them [31, 32] have the same form as in the ungauged case, given in [13], although the equations of motion are now those given above, which differ from those of the ungauged case by $g$-dependent terms.

Apart from the identities derived in ref. [13] we have found that, in the null case, there are additional identities that were overlooked in that reference. We will discuss them in section 3.2.

## 3. Supersymmetric configurations and solutions

Following the standard procedure, we assume that the KSEs

$$\mathfrak{D}_\mu \epsilon^i - \frac{1}{8\sqrt{3}} h_I F^{I\,\alpha\beta}\left(\gamma_{\mu\alpha\beta} - 4 g_{\mu\alpha}\gamma_\beta\right)\epsilon^i + \frac{1}{2\sqrt{3}} g \gamma_\mu \epsilon^j h^I P_{I\,j}{}^i = 0\,, \tag{3.1}$$

$$\left(\not{\mathfrak{D}}\phi^x - \frac{1}{2}h^x_I \not{F}^I\right)\epsilon^i + 2g\epsilon^j h^x_I P^I{}_j{}^i = 0\,, \tag{3.2}$$

$$f_X{}^{iA}\left(\not{\mathfrak{D}} q^X + \sqrt{3} g h^I k_I{}^X\right)\epsilon_i = 0\,, \tag{3.3}$$

admit at least one solution $\epsilon^i$ and we start deriving from them the equations satisfied by the tensor bilinears that can be constructed from the Killing spinor: the scalar $f$, the vector $V$ (both SU(2) singlets) and the three 2-forms $\Phi^r$, which form an SU(2)-triplet.

The fact that the Killing spinor satisfies eq. (3.1) leads to the following differential equations for the bilinears:

$$df = \frac{1}{\sqrt{3}} h_I i_V F^I\,, \tag{3.4}$$

$$\nabla_{(\mu} V_{\nu)} = 0\,, \tag{3.5}$$

$$dV = -\frac{2}{\sqrt{3}} f h_I F^I - \frac{1}{\sqrt{3}} h_I \star\left(F^I \wedge V\right) - \frac{2}{\sqrt{3}} g h^I \vec{P}_I \cdot \vec{\Phi}\,, \tag{3.6}$$

$$\mathfrak{D}_\alpha \vec{\Phi}_{\beta\gamma} = -\frac{1}{\sqrt{3}} h_I F^{I\,\rho\sigma} \left( g_{\rho[\beta} \star \vec{\Phi}_{\gamma]\alpha\sigma} - g_{\rho\alpha} \star \vec{\Phi}_{\beta\gamma\sigma} - \frac{1}{2} g_{\alpha[\beta} \star \vec{\Phi}_{\gamma]\rho\sigma} \right)$$
$$+ \frac{1}{\sqrt{3}} g h^I \left( \vec{P}_I \times (\star\vec{\Phi})_{\alpha\beta\gamma} + 2 g_{\alpha[\beta} V_{\gamma]} \vec{P}_I \right) , \tag{3.7}$$

where

$$\mathfrak{D}_\alpha \vec{\Phi}_{\beta\gamma} = \nabla_\alpha \vec{\Phi}_{\beta\gamma} + 2\vec{B}_\alpha \times \vec{\Phi}_{\beta\gamma} \,. \tag{3.8}$$

The differential equation for $\Phi^r$ (3.7) implies

$$d\Phi^r + 2\varepsilon^{rst} B^s \wedge \Phi^t = \sqrt{3} g h^I \epsilon^{rst} P^s{}_I \star \Phi^t \,. \tag{3.9}$$

The fact that the Killing spinor satisfies eqs. (3.2) and (3.3) leads to the following algebraic equations for the tensor bilinears:

$$V^\mu \mathfrak{D}_\mu \phi^x = 0 \,, \tag{3.10}$$
$$h^x_I F^I_{\alpha\beta} \vec{\Phi}^{\alpha\beta} = 4 g f h^x_I \vec{P}^I \,, \tag{3.11}$$
$$V^\mu \mathfrak{D}_\mu q^X = -\sqrt{3} g f h^I k_I{}^X \,, \tag{3.12}$$
$$f \mathfrak{D}_\mu \phi^x - h^x_I F^I{}_{\mu\nu} V^\nu = 0 \,, \tag{3.13}$$
$$\vec{\Phi}_{\mu\nu} \mathfrak{D}^\nu \phi^x + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta\gamma} h^x_I F^{I\,\nu\alpha} \vec{\Phi}^{\beta\gamma} = -2 g h^x_I \vec{P}^I V_\mu \,, \tag{3.14}$$
$$f \mathfrak{D}_\mu q^X + \Phi^r{}_\mu{}^\nu \mathfrak{D}_\nu q^Y J^r{}_Y{}^X = -\sqrt{3} g h^I k_I{}^X V_\mu \,. \tag{3.15}$$

We are now ready to extract consequences of these equations. To start with, eq. (3.5) says that $V$ is an isometry of the space-time metric. It is convenient to partially fix the $G$ gauge using the condition

$$i_V A^I + \sqrt{3} f h^I = 0 \,, \tag{3.16}$$

since then eqs. (3.12) and (3.10) become just

$$\mathcal{L}_V q^X = \mathcal{L}_V \phi^x = 0 \,, \tag{3.17}$$

after use of the explicit expression of the Killing vectors $k_I{}^x$ eq. (A.6). Then, in this gauge, the scalars $q^X, \phi^x$ and $f$ are independent of the coordinate adapted to the isometry (see eq. (3.4).

The Fierz identities relate the modulus of the vector bilinear $V^\mu$ to the scalar bilinear $f$: $V^2 = f^2$. This means that, as usual, $V^\mu$ can be timelike or null. We now consider separately the timelike ($f \neq 0$) and null ($f = 0$) cases.

## 3.1 The timelike case

### 3.1.1 The equations for the bilinears

By definition this is the case in which $V^\mu$ is timelike, $V^2 = f^2 > 0$. Introducing an adapted time coordinate $t$: $V = \partial_t$ the metric can be written in the same form as in the ungauged case:

$$ds^2 = f^2 (dt + \omega)^2 - f^{-1} h_{\underline{mn}} dx^m dx^n \,, \tag{3.18}$$

with $\omega$ and $h_{\underline{mn}}$ independent of time. As we mentioned in the previous section, in the (partially) fixed $G$-gauge ($A^I{}_t = -\sqrt{3} f h^I$) $f, \phi^x$ and $q^X$ are also time-independent.

The spatial metric $h_{\underline{mn}}$ is endowed with an *almost* quaternionic structure, $\Phi^r{}_m{}^n$. This is an algebraic property that only depends on the Fierz identities.

The next step is to obtain the form of the supersymmetric vector field strength from eqs. (3.4), (3.6), (3.11) and (3.13). In order to write the result it is convenient to split the gauge potential $A^I$ into an electric part, which is determined by the partial gauge fixing $A^I{}_t = -\sqrt{3} f h^I$ and a magnetic part $\hat{A}^I$ with only spatial components

$$A^I = -\sqrt{3} h^I e^0 + \hat{A}^I \,, \tag{3.19}$$

$$A^I{}_{\underline{m}} = \hat{A}^I{}_{\underline{m}} - \sqrt{3} f h^I \omega_{\underline{m}} \,. \tag{3.20}$$

Observe that, unlike the spatial components $A^I{}_{\underline{m}}$, the components $\hat{A}^I{}_{\underline{m}}$ are invariant under local shifts of the time coordinate: $t \to t + \delta t(x)$, $\omega \to \omega - d\delta t(x)$ which do not change the form of the metric and, in particular, leave the 4-dimensional metric $h_{\underline{mn}}$ invariant. It is the correct 4-dimensional potential in the Kaluza-Klein sense.

In terms of the new variables $\hat{A}^I$ the field strengths are given by

$$F^I = -\sqrt{3}\, \hat{\mathfrak{D}}(h^I e^0) + \hat{F}^I \,, \tag{3.21}$$

where $\hat{\mathfrak{D}}$ is the 4-dimensional spatial covariant derivative[6] with respect to $\hat{A}^I$ and $\hat{F}^I$ is the non-Abelian field strength of $\hat{A}^I$ and it is related to $\omega$ and the scalars by

$$h_I \hat{F}^{I+} = \frac{2}{\sqrt{3}} (f d\omega)^+ \,, \tag{3.22}$$

$$\hat{F}^{I-} = -2 g f^{-1} C^{IJK} h_J \vec{P}_K \cdot \vec{\Phi} \,. \tag{3.23}$$

$\tilde{F}^{I+}$ is related to the 2-forms called $\Theta^I$ in the ungauged case [7, 10, 13] by

$$\Theta^I = -\frac{1}{\sqrt{3}} \hat{F}^{I+} \,. \tag{3.24}$$

It is also convenient to introduce the spatial SU(2) connection $\hat{\vec{B}}$

$$\hat{\vec{B}} \equiv \vec{A} + \frac{1}{2} g \hat{A}^I \vec{P}_I \,, \tag{3.25}$$

$$\vec{B} = -\frac{\sqrt{3}}{2} h^I \vec{P}_I e^0 + \hat{\vec{B}} \,, \tag{3.26}$$

and extend the definition of $\hat{\mathfrak{D}}_{\underline{m}}$ as the spatial $G$- and SU(2)-covariant derivative made from the hatted connections $\hat{A}^I$ and $\hat{\vec{B}}$, including also the affine and spin connections of the base spatial manifold.

The eq. (3.15) is purely spatial in the timelike case and it becomes, in 4-dimensional notation[7]

$$\hat{\mathfrak{D}}_m q^X = \Phi^r{}_m{}^n \, \hat{\mathfrak{D}}_n q^Y \, J^r{}_Y{}^X \,. \tag{3.27}$$

[6] Strictly speaking the action of a 4-dimensional spatial covariant derivative on $e^0$ which contains $dt$ is not well-defined. It is understood that $\hat{\mathfrak{D}}(f dt) = \hat{\mathfrak{D}} f \wedge dt$.

[7] From now on spatial flat indices refer to the 4-dimensional spatial metric $h_{\underline{mn}}$.

We notice that this equation, even though it is written in terms of covariant derivatives, imposes no integrability condition on the gauge connections. That is, as equation for $q^X$ it has always *local* solution for any given vector fields $\hat{A}^I$.

Projecting this equation along the Killing vectors $k_I$ yields an important relation,

$$k_{I\,X}\hat{\mathfrak{D}}_m q^X = -2\vec{\Phi}_m{}^n\hat{\mathfrak{D}}_n\vec{P}_I\,. \tag{3.28}$$

This projection is the one which appears in the Maxwell equations (2.6).

Let us study the differential equations for the two-forms $\vec{\Phi}$. The projection of eq. (3.9) along $V$ says that they are time-independent in the gauge (3.16):

$$\partial_t\vec{\Phi}_{mn} = 0\,. \tag{3.29}$$

The components of eq. (3.7) can be explicitly evaluated using the 5-dimensional metric eq. (3.18) and the expression for the field strengths eq. (3.21). Only the spatial components of the 5-dimensional covariant derivative give new information:

$$\hat{\mathfrak{D}}_m\vec{\Phi}_{np} = 0\,. \tag{3.30}$$

This is a condition for the anti-self-dual part of the spin connection $\xi$ of the base spatial manifold. Indeed we can solve for $\xi^-$ in an arbitrary frame and SU(2) gauge:

$$\xi^-{}_{mnp} = -\hat{\vec{B}}_m\cdot\vec{\Phi}_{np} - \frac{1}{4}\partial_m\vec{\Phi}_{nq}\cdot\vec{\Phi}_{qp}\,, \tag{3.31}$$

where we have used the (Fierz) identity

$$\vec{\Phi}_{mn}\cdot\vec{\Phi}_{pq} = \delta_{mp}\delta_{nq} - \delta_{mq}\delta_{np} - \epsilon_{mnpq}\,. \tag{3.32}$$

The meaning of relation (3.31) becomes clearer in a frame and SU(2) gauge in which the $\vec{\Phi}$s are constant: the SU(2) connection $\hat{\vec{B}}$ is embedded into the anti-self-dual part of the spin connection of the base manifold. The same happenend in the ungauged case [13] and, again, this embedding requires the action of the SU(2) generators in the fundamental and spinorial representation on spinors to be identical, i.e.

$$\epsilon^j\; i\vec{\sigma}_j{}^i \;=\; \frac{1}{4}\vec{\mathrm{J}}_{mn}\gamma^{mn}\;\epsilon^i\,, \tag{3.33}$$

and these conditions will appear as projectors

$$\Pi^{r\,\pm}{}_i{}^j \;=\; \frac{1}{2}\left[\;\delta\;\pm\;\frac{i}{4}\;\not{J}^{(r)}\sigma^{(r)}\right]_i{}^j\,, \tag{3.34}$$

acting on the Killing spinors.

It is interesting to study the integrability condition of eq. (3.30), which is

$$\left[\frac{1}{4}R^-{}_{mnkl}\vec{\Phi}^{kl} + \vec{R}_{mn}(\hat{\vec{B}})\right]\times\vec{\Phi}_{pq} = 0\,, \tag{3.35}$$



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where $\vec{R}_{mn}(\hat{\vec{B}})$ is the curvature of $\hat{\vec{B}}$, which is given by

$$\vec{R}_{mn}(\hat{\vec{B}}) = \hat{\mathfrak{D}}_m q^X \hat{\mathfrak{D}}_n q^Y \vec{R}_{XY}(\vec{\omega}) + \frac{1}{2} g \hat{F}^I_{mn} \vec{P}_I = -\frac{1}{4} \hat{\mathfrak{D}}_m q^X \hat{\mathfrak{D}}_n q^Y \vec{J}_{XY} + \frac{1}{2} g \hat{F}^I{}_{mn} \vec{P}_I \,, \quad (3.36)$$

hence the integrability condition yields

$$R^-{}_{mnkl} \vec{\Phi}^{kl} - \hat{\mathfrak{D}}_m q^X \hat{\mathfrak{D}}_n q^Y \vec{J}_{XY} + 2g \hat{F}^I{}_{mn} \vec{P}_I = 0 \,. \quad (3.37)$$

We stress that this condition is equivalent to eq. (3.31).

Now if we contract this expression with $\vec{\Phi}^{pn}$ we can compare it with eq. (A.26) and doing so we obtain an expression involving the Ricci tensor of the spatial metric $h_{\underline{mn}}$

$$R_{mn}(h) = -\frac{1}{2} \hat{\mathfrak{D}}_m q^X \hat{\mathfrak{D}}_n q^Y g_{XY} + 2g^2 f^{-1} C^{IJK} h_I \vec{P}_J \cdot \vec{P}_K \delta_{mn} + g \hat{F}^{I+}{}_{mp} \vec{\Phi}_{pn} \cdot \vec{P}_I \,, \quad (3.38)$$

where we have used again the identity (3.32), and consequently the Ricci scalar

$$R(h) = -\frac{1}{2} \hat{\mathfrak{D}}_m q^X \hat{\mathfrak{D}}_m q^Y g_{XY} + 8g^2 f^{-1} C^{IJK} h_I \vec{P}_J \cdot \vec{P}_K \,. \quad (3.39)$$

In the ungauged case the eq. (3.38) says that the Ricci tensor of the spatial metric $h_{\underline{mn}}$ is proportional to the induced metric

$$R_{mn}(h) = -\frac{1}{2} \partial_m q^X \partial_n q^Y g_{XY} \,. \quad (3.40)$$

On the other hand in the gauged case we can solve the eq. (3.39) for $f$,

$$f = (8g^2 C^{IJK} h_I \vec{P}_J \cdot \vec{P}_K) / (R(h) + \frac{1}{2} \hat{\mathfrak{D}}_m q^X \hat{\mathfrak{D}}_m q^Y g_{XY}) \,. \quad (3.41)$$

### 3.1.2 Solving the Killing spinor equations

We are now going to prove that the necessary conditions for having unbroken supersymmetry that we have derived in the previous section are also sufficient. Thus, we are going to assume that we have a configuration with a metric of the form eq. (3.18), a non-Abelian gauge potential of the form eq. (3.19) with a field strength of the form eq. (3.21) satisfying eqs. (3.22) and (3.23), and hyperscalars such that eqs. (3.27) and (3.31) are satisfied.

Substituting these expressions in the KSE associated to the gaugino SUSY transformation rule eq. (3.2), and expressing all terms in 4-dimensional language we get

$$f^{1/2} \left( 2 \, \hat{\not{\mathfrak{D}}} \phi^x - \frac{\sqrt{3}}{2} f^{1/2} h^x_I \; \hat{\not{F}}^{I+} \right) R^- \epsilon^i + 2g h^x_I \vec{P}^I \cdot \left( i \vec{\sigma}_j{}^i - \frac{1}{4} \; \vec{\not{\Phi}} \delta_j{}^i \right) \epsilon^j = 0 \,. \quad (3.42)$$

where

$$R^\pm \equiv \frac{1}{2} \left( 1 \pm \gamma^0 \right) \,, \qquad \Pi^{r\pm}{}_j{}^i \equiv \frac{1}{2} \left( \delta \;\pm\; \frac{i}{4} \; \not{\Phi}^{(r)} \sigma^{(r)} \right)_j{}^i \,. \quad (3.43)$$

The projections

$$\vec{\Pi}^+{}_j{}^i \epsilon^j = 0 \,, \qquad R^- \epsilon^i = 0 \,, \quad (3.44)$$



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are sufficient to solve it. All of them are necessary in the general case but in particular cases in which the coefficients of the projectors in the above and following equations vanish, only some of them may be necessary. The discussion is entirely analogous to that of the ungauged case [13].

Substituting now in eq. (3.3) we get

$$f_X{}^{iA}\{f^{1/2}\,\hat{\not{\mathfrak{D}}}q^X\epsilon_i+2\sqrt{3}gh^Ik_I{}^Xf_X{}^{iA}R^-\}\epsilon_i=0\,. \tag{3.45}$$

The last term vanishes with the second projection of eqs. (3.44). On the other hand, from eq. (3.27) we can derive the identity

$$f_X{}^{iA}\,\hat{\not{\mathfrak{D}}}q^X R^+=-f_X{}^{jA}\,\hat{\not{\mathfrak{D}}}q^X\sum_r\left(\Pi^{r+}-\Pi^{r-}\right)_j{}^i\,. \tag{3.46}$$

Acting on $\epsilon_i$ and imposing again the projections (3.44) we see that it leads to

$$f_X{}^{iA}\,\hat{\not{\mathfrak{D}}}q^X\epsilon_i=-3f_X{}^{iA}\,\hat{\not{\mathfrak{D}}}q^X\epsilon_i\qquad\Rightarrow\qquad f_X{}^{iA}\,\hat{\not{\mathfrak{D}}}q^X\epsilon_i=0\,. \tag{3.47}$$

Hence the hyperino KSE (3.45) is also solved.

Finally, the spatial components of the same equation take, using $R^-\epsilon^i=0$, the form

$$\nabla_m\eta^i+\eta^jC_{m\,j}{}^i=0\,,\qquad\quad \eta^i\equiv f^{-1/2}\epsilon^i\,. \tag{3.48}$$

Using the relation (3.31) and the projections, it becomes

$$\partial_m\eta^i+\frac{1}{16}\partial_m\,\not{\Phi}_j{}^i\eta^j=0\,, \tag{3.49}$$

where $\Phi_i{}^j=i\vec{\sigma}_i{}^j\cdot\vec{\Phi}$.

The solution of this equation is given in terms of the path-ordered exponential

$$\eta^i(x,x_0)=P\exp\left(-\frac{1}{16}\int_{x_0}^{x}dx_1^{\underline{m}}\partial_{\underline{m}}\,\not{\Phi}_j{}^i(x_1)\right)\eta_0^j\,, \tag{3.50}$$

where $\eta_0^i$ is a constant spinor, or in a frame and SU(2) gauge where $\vec{\Phi}$ is constant, it is just the constant spinor $\eta_0^i$.

The analysis of the amount of unbroken supersymmetry is identical to that of the ungauged case [13].

### 3.1.3 Supersymmetric solutions

As we discussed at the end of section 2, the KSIs of the gauged theories have the same form as those of the ungauged ones, which are given in ref. [13]. There it was proven that timelike supersymmetric configurations solve all the equations of motions if they solve the Maxwell equations. We are now going to impose those equations on the supersymmetric configurations. It is possible to show that the Bianchi identities imply the spatial components of the Maxwell equations for supersymmetric configurations using eq. (3.28)

$$\mathcal{E}_I{}^m=2C_{IJK}h^J(\star\mathfrak{D}F^K)^{0m}\,. \tag{3.51}$$

Thus we only need to impose the time component of the Maxwell equations on the supersymmetric configurations. This equation takes the form

$$\hat{\mathfrak{D}}^2 (h_I/f) - \frac{1}{12} C_{IJK} \hat{F}^J \cdot \hat{F}^K + \frac{2}{\sqrt{3}} C_{IJK} h^J \hat{F}^K \cdot G^- + 2g^2 f^{-2} g_{XY} k_I{}^X k_J{}^Y h^J = 0\,, \quad (3.52)$$

where

$$G \equiv f d\omega\,. \quad (3.53)$$

This is the only equation that has to be solved if we have a configuration which we know is supersymmetric and admits a gauge potential. It differs from that of the ungauged case in the gauge-covariant derivatives and in the last two terms. The first of these is implicitly first-order in $g$, due to eq. (3.23) and the second one is manifestly second-order in $g$.

Constructing a supersymmetric configuration is, now, considerably more complex than in the ungauged or Abelian-gauged cases and it seems not possible to give an algorithm which automatically returns supersymmetric configurations. At any rate, a possible recipe to construct a supersymmetric configuration of a given $N = 1, d = 5$ gauged supergravity theory is the following.

(i) The objects that have to be chosen are

- (i.i) The 4-dimensional spatial metric $h_{\underline{mn}}(x)$ and an almost complex structure $\vec{\Phi}_{mn}$. The former determines the anti-selfdual part of its spin connection: $\xi^-{}_{mnp}$.
- (i.ii) A spatial 1-form $\omega_{\underline{m}}$.
- (i.iii) The $4n_H$ hyperscalar mappings $q^X(x)$ from the 4-dimensional spatial manifold to the quaternionic-Kähler manifold. They determine the (pullbacks of) the momentum map[8] $\vec{P}_I$ and the SU(2) connection $\vec{A}_{\underline{m}} = \partial_{\underline{m}} q^X \vec{\omega}_X$
- (i.iv) A spatial gauge potential $\hat{A}^I{}_{\underline{m}}$. It determines the spatial gauge field strength $\hat{F}^I{}_{\underline{mn}}$ and, together with the pullback of the SU(2) connection $\vec{A}_{\underline{m}}$ and the momentum map, it determines the spatial SU(2) connection $\hat{\vec{B}}$ whose definition we rewrite here for convenience:

$$\hat{\vec{B}} \equiv \vec{A} + \frac{1}{2} g \hat{A}^I \vec{P}_I\,.$$

- (i.v) $\bar{n} = n_V + 1$ scalar functions $h_I/f$. They determine, upon use of the constraint $C_{IJK} h^I h^J h^L = 1$ the $n_V$ scalars $\phi^x$ and the metric function $f$.[9] Together with $\hat{A}^I{}_{\underline{m}}$ and $\omega_{\underline{m}}$ they give the full 5-dimensional gauge potential $A^I{}_\mu$

$$A^I = -\sqrt{3} h^I e^0 + \hat{A}^I\,.$$

(ii) These objects now have to satisfy the following equations:

[8] If $n_H = 0$ they are constant Fayet-Iliopoulos terms as explained in footnote 13.

[9] One can also use eq. (3.41) to determine $f$.



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(ii.i) Eq. (3.31) that embeds the spatial SU(2) connection $\hat{\vec{B}}$ into the spin connection of the base spatial manifold.

$$\xi^-{}_{mnp} = -\hat{\vec{B}}_m \cdot \vec{\Phi}_{np} - \frac{1}{4}\partial_m \vec{\Phi}_{nq} \cdot \vec{\Phi}_{qp} \,,$$

(ii.ii) Eq. (3.27) that characterizes the hyperscalar mappings

$$\hat{\mathfrak{D}}_m q^X = \Phi^r{}_m{}^n \; \hat{\mathfrak{D}}_n q^Y \; J^r{}_Y{}^X \,.$$

(ii.iii) Eqs. (3.22) and (3.23)

$$h_I \hat{F}^{I+} = \frac{2}{\sqrt{3}}(fd\omega)^+ \,, \tag{3.54}$$

$$\hat{F}^{I-} = -2gf^{-1}C^{IJK}h_J \vec{P}_K \cdot \vec{\Phi} \,. \tag{3.55}$$

(ii.iv) Finally, eq. (3.52)

$$\hat{\mathfrak{D}}^2 \left(h_I/f\right) - \frac{1}{12}C_{IJK}\hat{F}^J \cdot \hat{F}^K + \frac{2}{\sqrt{3}}C_{IJK}h^J \hat{F}^K \cdot G^- + 2g^2 f^{-2} g_{XY} k_I{}^X k_J{}^Y h^J = 0 \,.$$

As we see, finding supersymmetric solutions remains a difficult problem and we leave for future work the construction of explicit examples [23].

## 3.2 The null case

### 3.2.1 The equations for the bilinears

As usual, we denote the null Killing vector by $l^\mu$ and choose null coordinates $u$ and $v$ such that

$$l_\mu dx^\mu = f du \,, \qquad l^\mu \partial_\mu = \partial_{\underline{v}} \,, \tag{3.56}$$

where $f$ may depend on $u$ but not on $v$. The metric can be put in the form

$$ds^2 = 2f du(dv + H du + \omega) - f^{-2}\gamma_{\underline{rs}} dx^r dx^s \,, \tag{3.57}$$

where $r,s,t = 1,2,3$ and the 3-dimensional spatial metric $\gamma_{\underline{rs}}$ may also depend on $u$ but not on $v$. With these coordinates the partial gauge fixing (3.16), for $g \neq 0$, becomes just $A^I_{\underline{v}} = 0$. Eqs. (3.10) and (3.17) state that the scalars are $v$-independent.

In the null case Fierz identities imply that the 2-forms bilinears $\Phi^r$ are given by

$$\Phi^r = du \wedge v^r \,, \tag{3.58}$$

where the $v^r$ are 1-forms that can be used as Dreibein for the spatial metric $\gamma_{\underline{rs}}$.

We decompose the gauge potential as

$$A^I = A^I{}_{\underline{u}} du + \hat{A}^I \,, \tag{3.59}$$

where $\hat{A}$ is a spatial one-form. Under a $u$-independent $G$-transformation $\hat{A}^I$ transforms as a gauge connection whereas $A^I{}_{\underline{u}}$ transforms homogeneously. We denote by $\hat{\mathfrak{D}}$ the spatial



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covariant derivative made with the three-dimensional affine and spin connections and the gauge connection $\hat{A}^I$.

Eq. (3.9) becomes

$$du \wedge \left[dv^r - \left(2\varepsilon^{rst}\hat{B}^t + \sqrt{3}gf^{-1}h^I P_I^s v^r\right) \wedge v^s\right] = 0\,, \tag{3.60}$$

where, again, $\hat{B}^t$ is $B^t$ with $A^I$ replaced by $\hat{A}^I$. This equation relates the the tridimensional spin connection (computed for constant $u$) to the spatial components of the pullback of the SU(2):

$$\varpi^{rs} = 2\varepsilon^{rst}\hat{B}^t - 2\sqrt{3}gf^{-1}h^I P_I^{[r}v^{s]}\,. \tag{3.61}$$

Substituting the 2-forms we found into eq. (3.15) we arrive at

$$\hat{\mathfrak{D}}_r q^X J^r{}_X{}^Y = \sqrt{3}gf^{-1}h^I k_I{}^Y\,, \tag{3.62}$$

which is the condition that must be satisfied by the mappings $q^X$ in order to have supersymmetry.

Let us now determine the vector field strengths: eqs. (3.4) and (3.13) lead to

$$l^\mu F^I{}_{\mu\nu} = 0\,, \tag{3.63}$$

which implies that the field strengths have the general form

$$F^I = F^I{}_{+r}e^+\wedge e^r + \frac{1}{2}f^2F^I{}_{rs}e^r\wedge e^s = F^I{}_{+r}du\wedge v^r + \frac{1}{2}F^I{}_{rs}v^r\wedge v^s \equiv F^I{}_{+r}du\wedge v^r + \hat{F}^I\,. \tag{3.64}$$

From eq. (3.6) we get

$$h_I\hat{F}^I = \sqrt{3}\hat{\star}\hat{d}f^{-1} + 2gf^{-2}h^I\hat{\star}\hat{P}_I\,, \tag{3.65}$$

where $\hat{P}_I$ is the spatial 1-form

$$\hat{P}_I = P^r{}_I v^r\,. \tag{3.66}$$

On the other hand eq. (3.14) yields

$$h_I^x\hat{F}^I = -f^{-1}\hat{\star}\hat{\mathfrak{D}}\phi^x + 2gf^{-2}h_I^x\hat{\star}\hat{P}^I\,, \tag{3.67}$$

which, together with the previous equation and the definition of $h_I^x$ give

$$\hat{\star}\hat{F}^I = \sqrt{3}\hat{\mathfrak{D}}(h^I/f) + 2gf^{-2}\hat{P}^I\,. \tag{3.68}$$

From the $++r$ components of eq. (3.7) we get

$$h_IF^I{}_{+r} = -\frac{1}{\sqrt{3}}f^2(\hat{\star}F)_r\,, \tag{3.69}$$

where

$$F = \hat{d}\omega\,. \tag{3.70}$$

The components $h_I^xF^I{}_{+r}$ are not determined by supersymmetry and we parametrize them by 1-forms $\psi^I$ satisfying $h_I\psi^I = 0$. In conclusion, the vector field strengths must take the general form

$$F^I = (\frac{1}{\sqrt{3}}f^2h^I\hat{\star}F - \psi^I)\wedge du + \sqrt{3}\hat{\star}\left[\hat{\mathfrak{D}}(h^I/f) + \frac{2}{\sqrt{3}}gf^{-2}\hat{P}^I\right]\,. \tag{3.71}$$



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### 3.2.2 Solving the Killing spinor equations

It is not difficult to check that, for field configurations with metric of the form eq. (3.57), vector field strengths of the form eq. (3.71) and hyperscalars satisfying eq. (3.62), the KSEs admit solutions which are constant spinors satisfying the constraint

$$\gamma^{+}\epsilon^{i} = 0\,, \tag{3.72}$$

and a constraint of the form

$$\Pi^{r}\epsilon = 0\,, \tag{3.73}$$

for every $r$ for which $\hat{B}^{r}$ and $gfh^{I}P_{I}^{r}$ do not vanish, where $\Pi^{r}$ is the projector

$$\Pi^{r}{}_{i}{}^{j} \;=\; \frac{1}{2}\left(\delta - i\gamma^{(r)}\sigma^{(r)}\right)_{i}{}^{j} \quad ; \quad \Pi^{r\,2} \;=\; \Pi^{r} \quad ; \quad [\,\Pi^{r}\,,\,\Pi^{s}\,] \;=\; 0\,. \tag{3.74}$$

Each of these projections breaks/preserves one half of the supersymmetries. In the general case one must impose the three projections given in eq. (3.73). It should be noted that in this case the projection (3.72) is already implied by the whole system of projections (3.73). Thus we have that the general supersymmetric configurations preserve 1/8 of the supersymmetries.

As it happened in ref. [13] consistency with the space-independence of the Killing spinors requires the $u$-component of $B$ to have the form

$$v_{[r}{}^{\underline{r}}\partial_{\underline{u}}v_{s]\underline{r}} = -2\varepsilon_{rst}B^{t}{}_{\underline{u}}\,. \tag{3.75}$$

### 3.2.3 Equations of motion

We now want to impose the equations of motion on the supersymmetric configurations that we have identified. On supersymmetric configurations only a few equations of motion are independent, since they are related by the Killing Spinor Identities (KSIs) [31, 32] which, as discussed in section 2, for these theories were computed in ref. [13]. A few KSIs were overlooked, however, in the reference. They reduce considerably the number of independent equations to be checked and we start by computing them.

**Additional KSIs.** According to eq. (3.58) the only non-vanishing components of the 2-forms $\Phi^{r}$ are

$$\Phi^{r\,s-} = \delta^{rs}\,. \tag{3.76}$$

We can use this result to find additional constraints in the equations of motion from the KSIs [13]

$$\left[\left(\mathcal{E}_{bc} + \frac{\sqrt{3}}{2}h_{I}\star\mathcal{B}^{I}{}_{bc}\right)\gamma^{c} + \frac{\sqrt{3}}{2}h^{I}\mathcal{E}_{I\,b}\right]\epsilon^{i} = 0\,, \tag{3.77}$$

$$\left[\mathcal{E}_{x} - h^{I}_{x}\left(\not{\mathcal{E}}_{I} + \frac{1}{6}a_{IJ}\,\not{\mathcal{B}}^{J}\right)\right]\epsilon^{i} = 0\,. \tag{3.78}$$

Acting with $(\sigma^{r})^{j}{}_{i}\bar{\epsilon}_{j}\gamma^{a}$ on eq. (3.77), we get

$$\left(\mathcal{E}_{bc} + \frac{\sqrt{3}}{2}h_{I}\star\mathcal{B}^{I}{}_{bc}\right)\Phi^{r\,ac} = 0\,. \tag{3.79}$$

Taking $a = -, r$ we get, respectively

$$\mathcal{E}_{br} = -\frac{\sqrt{3}}{2} h_I \star \mathcal{B}^I{}_{br}\,, \tag{3.80}$$

$$\mathcal{E}_{b-} = -\frac{\sqrt{3}}{2} h_I \star \mathcal{B}^I{}_{b-}\,. \tag{3.81}$$

The second identity was already found in [13]. The symmetry of the l.h.s. and the anti-symmetry of the r.h.s. of both identities and the combination of both implies

$$\mathcal{E}_{r-} = h_I \star \mathcal{B}^I{}_{r-} = 0\,, \tag{3.82}$$

$$\mathcal{E}_{rs} = h_I \star \mathcal{B}^I{}_{rs} = 0\,. \tag{3.83}$$

Eqs. (3.80)–(3.83) leave us with only three non-vanishing components of the Einstein equations, namely $\mathcal{E}_{++}, \mathcal{E}_{+-}, \mathcal{E}_{+t}$, of which the last two are proportional to components of the Bianchi identities. Thus, the only independent component of the Einstein equation is $\mathcal{E}_{++}$.

Acting now with $(\sigma^r)^j{}_i \bar{\epsilon}_j$ on eq. (3.78), we get

$$h_{Ix} \star \mathcal{B}^I{}_{ab} \Phi^{r\,ab} = 0\,, \quad \Rightarrow \quad h_{Ix} \star \mathcal{B}^I{}_{-r} = 0\,, \tag{3.84}$$

which, together with eq. (3.82) leads to

$$\star \mathcal{B}^I{}_{-r} = 0\,. \tag{3.85}$$

Acting with $(\sigma^r)^j{}_i \bar{\epsilon}_j \gamma^a$ on eq. (3.78), we get

$$h^I_x \mathcal{E}_{I-} = 0\,, \tag{3.86}$$

$$h^I_x \mathcal{E}_{Ir} = \frac{1}{2} h_{Ix} \varepsilon_{rst} \star \mathcal{B}^I{}_{st}\,, \tag{3.87}$$

which, together with $h^I \mathcal{E}_{I\,\mu} = 0$ (proven in ref. [13]) imply

$$\mathcal{E}_{I-} = 0\,. \tag{3.88}$$

The only independent components of the Maxwell equations are $h^I_x \mathcal{E}_{I+}$.

Summarizing, unbroken supersymmetry implies that the only non-automatically vanishing components of the Einstein and Maxwell equations and Bianchi identities are $\mathcal{E}_{++}, \mathcal{E}_{+-}, \mathcal{E}_{+r}$, $\mathcal{B}^I{}_{+-}, \mathcal{B}^I{}_{+r}, \mathcal{B}^I{}_{rs}$ and $\mathcal{E}_{I+}, \mathcal{E}_{Ir}$. The scalar equations of motion are always automatically satisfied. If the Bianchi identities are satisfied, as they must in this case,[10] only $\mathcal{E}_{++}$ and $\mathcal{E}_{I+}$ need to be explicitly checked.

[10] In the non-Abelian case that we are considering here the knowledge of the gauge potential is necessary in order to construct a supersymmetric configuration, which is our starting point, and the Bianchi identities are always assumed to be satisfied. Nevertheless, since the gauge field strength is related to other fields, the Bianchi identities lead to constraints on the other fields.



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**Independent equations of motion.** Let us start with the Bianchi identities. Using the decomposition of the potential eq. (3.59) we obtain from the expression for the gauge field strength eq. (3.71) two equations:

$$\hat{F}^I = \sqrt{3}\hat{\star}\left[\hat{\mathfrak{D}}(h^I/f) + \frac{2}{\sqrt{3}} g f^{-2}\hat{P}^I\right], \tag{3.89}$$

$$\hat{\mathfrak{D}}A^I{}_{\underline{u}} - \partial_{\underline{u}}\hat{A}^I = \frac{1}{\sqrt{3}} f^2 h^I \hat{\star} F - \psi^I . \tag{3.90}$$

The Bianchi identity of the first equation leads to

$$\hat{\mathfrak{D}}\hat{\star}\hat{\mathfrak{D}}(h^I/f) + \frac{2}{\sqrt{3}} g\hat{\mathfrak{D}}(f^{-2}\hat{\star}\hat{P}^I) = 0 . \tag{3.91}$$

The constraint $h_I\psi^I = 0$ and the second equation imply

$$\frac{1}{\sqrt{3}} f^2\hat{\star}F - h_I\hat{\mathfrak{D}}A^I{}_{\underline{u}} + h_I\partial_{\underline{u}}\hat{A}^I = 0 , \tag{3.92}$$

which can be taken as the equation defining $\omega$. Having $\omega$ and the potentials eq. (3.90) determines $\psi^I$:

$$\psi^I = \frac{1}{\sqrt{3}} f^2 h^I\hat{\star}F - \hat{\mathfrak{D}}A^I{}_{\underline{u}} + \partial_{\underline{u}}\hat{A}^I . \tag{3.93}$$

Apart from these equations we have to impose the Maxwell equations, which, in differential form language take the form

$$4\star\mathcal{E}_I = -\mathfrak{D}\star\left(a_{IJ}F^J\right) + \frac{1}{\sqrt{3}} C_{IJK}F^J\wedge F^K + g\star\left(k_{I\,x}\mathfrak{D}\phi^x + k_{I\,X}\mathfrak{D}q^X\right) . \tag{3.94}$$

Substituting the gauge field strength and operating we get

$$\begin{aligned} 4\star\mathcal{E}_I = du\wedge\Big\{ & g\left[\sqrt{3}f_{IJ}{}^K\hat{F}^J h_K f - 2\hat{\mathfrak{D}}\hat{P}_I - \hat{\star}(k_{I\,x}\hat{\mathfrak{D}}\phi^x + k_{I\,X}\hat{\mathfrak{D}}q^X)\right]\wedge(dv+\omega) \\ & -\sqrt{3}\left[\hat{\mathfrak{D}}(h_I f) - \frac{2}{\sqrt{3}} g\hat{P}_I\right]\wedge F + \frac{1}{\sqrt{3}}\hat{\mathfrak{D}}(f h_I)\wedge F - \hat{\mathfrak{D}}(f^{-1}\hat{\star}\psi_I) \\ & -g f^{-3}\hat{\star}\left(k_{I\,x}\mathfrak{D}_{\underline{u}}\phi^x + k_{I\,X}\mathfrak{D}_{\underline{u}}q^X\right) - \frac{2}{\sqrt{3}} C_{IJK}\left(\frac{1}{\sqrt{3}} f^2 h^J\hat{\star}F - \psi^J\right)\wedge\hat{F}^K\Big\} . \end{aligned} \tag{3.95}$$

The first line contributes to $\mathcal{E}_{I\,r}$ and it can be checked (thorugh a long and painful calculation) that it vanishes automatically for supersymmetric configurations, as it should according to the KSIs, while the other two lines contribute to $\mathcal{E}_{I\,+}$.

The Maxwell equations, then, simplify and take the form

$$\begin{aligned} 4\star\mathcal{E}_I = du\wedge\Big\{ & \left[-\sqrt{3}f\hat{\mathfrak{D}}(h_I) + 2g\hat{P}_I - \frac{4}{3} g C_{IJK}h^J\hat{P}^K\right]\wedge F - \hat{\mathfrak{D}}(\hat{\star}\psi_I/f) \\ & + \frac{2}{\sqrt{3}} C_{IJK}\psi^J\wedge\hat{F}^K - g f^{-3}\hat{\star}\left(k_{I\,x}\mathfrak{D}_{\underline{u}}\phi^x + k_{I\,X}\mathfrak{D}_{\underline{u}}q^X\right)\Big\} . \end{aligned} \tag{3.96}$$

As implied by the KSIs only the $\mathcal{E}_{I+}$ component is not automatically satisfied and has to be explicitly imposed in order to get classical solutions. It can be also be checked that $h^I\mathcal{E}_{I+} = 0$ (as it is implied by the KSIs) up to terms that are proportional to $d^2\omega$.

The same fact can be described in a slightly different way: the integrability condition of the $\omega$ equation ($d^2\omega = 0$) is satisfied if supersymmetry is unbroken and the KSI $h^I\mathcal{E}_{I+} = 0$ is satisfied. In general, as first pointed out in refs. [33, 34] there will be singular points at which this will not happen. These points give rise to physical singularities in the metric and, therefore, they should not be allowed in meaningful solutions. This requirement translates into constraints on charges and asymptotic values of the moduli. It can be argued that this requirement is equivalent to the requirement of having supersymmetry unbroken everywhere (and the KSIs satisfied everywhere) [35, 36].

In order to write the equations of motion in a simple form it is convenient to define some new variables:

$$h^I/f \equiv K^I\,, \qquad\qquad f^{-3} = C_{IJK}K^IK^JK^K\,, \tag{3.97}$$

$$L_I \equiv C_{IJK}K^JA^K{}_{\underline{u}}\,, \tag{3.98}$$

$$N \equiv H + \frac{1}{2}L_IA^I{}_{\underline{u}}\,. \tag{3.99}$$

Observe that $\frac{1}{\sqrt{3}}\hat{A}^I$ and $-A^I{}_{\underline{u}}$ coincide, respectively, with what was called $\alpha^I$ and $M^I$ in the ungauged case, in ref. [13].

Using these variables and eq. (3.93), the Maxwell equation can be put into the form

$$\begin{aligned}4\star\mathcal{E}_I = -2du\wedge\Big\{&\hat{\mathfrak{D}}\hat{\star}\hat{\mathfrak{D}}L_I - g\hat{P}_I\wedge F + \frac{2}{\sqrt{3}}gC_{IJK}\hat{\mathfrak{D}}\hat{\star}(f^{-2}A^J{}_{\underline{u}}\hat{P}^K)\\ &-gC_{IJK}\left[\hat{\mathfrak{D}}\hat{\star}(K^J\partial_{\underline{u}}\hat{A}^K) + \left(\hat{\mathfrak{D}}K^J + \frac{2}{\sqrt{3}}g\hat{\star}\hat{P}^J\right)\wedge\hat{\star}\partial_{\underline{u}}\hat{A}^K\right]\\ &+\frac{1}{2}gf^{-3}\hat{\star}\left(k_{I\,x}\mathfrak{D}_{\underline{u}}\phi^x + k_{I\,X}\mathfrak{D}_{\underline{u}}q^X\right)\Big\}\,.\end{aligned} \tag{3.100}$$

This equation is gauge-invariant, in particular, under $u$-dependent $G$-gauge transformations that act on $\hat{A}^I$, $A^I_{\underline{u}}$, $L_I$ and the bosonic scalars. This fact can be used to partially fix the $G$ gauge, as done in ref. [13], leaving a much simpler equation which is still covariant under $u$-independent $G$ gauge transformations.

The 1-form $\omega$ is determined by eq. (3.92) only up to total derivatives which correspond to shifts in the coordinate $v$. This transformation must be accompanied with a shift in $H$ (or $N$). We can use this freedom to impose a condition on (basically, the $u$-dependence of) $\omega$:

$$\begin{aligned}\nabla_r(\dot{\omega})_r + 3(\dot{\omega})_r\partial_r\log f = &-\frac{1}{2}f^{-3}(\ddot{\gamma})_{rr} - \frac{1}{4}f^{-3}(\dot{\gamma})^2 + \frac{3}{2}f^{-4}\dot{f}(\dot{\gamma})_{rr}\\ &+3f^{-3}[\partial^2_{\underline{u}}\log f - 2(\partial_{\underline{u}}\log f)^2]\\ &-\frac{1}{2}f^{-3}\left[g_{xy}(\dot{\phi}^x\dot{\phi}^y + 2g\dot{q}^xA^I{}_{\underline{u}}k_I{}^y) + g_{XY}(\dot{q}^X\dot{q}^Y + 2g\dot{q}^XA^I{}_{\underline{u}}k_I{}^Y)\right]\\ &+C_{IJK}K^I\left[(\partial_{\underline{u}}\hat{A}^J)_r(\partial_{\underline{u}}\hat{A}^K)_r - 2\hat{\mathfrak{D}}_rA^J{}_{\underline{u}}(\partial_{\underline{u}}\hat{A}^K)_r\right]\,.\end{aligned} \tag{3.101}$$

After performing these steps, the $\mathcal{E}_{++}$ component of the Einstein equations becomes

$$\begin{aligned}-f^{-1}\mathcal{E}_{++} = &\nabla^2N + \frac{1}{\sqrt{3}}g\hat{\mathfrak{D}}_r(f^{-2}C_{IJK}P^I_rA^J_{\underline{u}}A^K{}_{\underline{u}})\\ &+\frac{1}{2}gf^{-3}A^I{}_{\underline{u}}A^J{}_{\underline{u}}(g_{xy}k_I{}^xk_J{}^y + g_{XY}k_I{}^Xk_J{}^Y)\,.\end{aligned} \tag{3.102}$$

Let us summarize the results of this section by giving the "recipe" to build supersymmetric solutions in the null class.

(i) The objects that have to be chosen are

(i.i) A spatial 3-dimensional metric $\gamma_{\underline{rs}}$ and Dreibein basis $v^r$ both of which may depend on the null coordinate $u$. This determines the 3-dimensional spin connection $\varpi^{rs}$.

(i.ii) The $4n_H$ hyperscalar $u$-dependent mappings $q^X(x,u)$ from the 3-dimensional spatial manifold to the quaternionic-Kähler manifold. They determine the (pullbacks of) the momentum map $\vec{P}_I$ and the SU(2) connection $\vec{A}_{\underline{r}} = \partial_{\underline{r}} q^X \vec{\omega}_X$ and $\vec{A}_{\underline{u}} = \partial_{\underline{u}} q^X \vec{\omega}_X$

(i.iii) A gauge connection 1-form $A^I$ with vanishing $v$ component. This determines its spatial and null parts $\hat{A}^I$ and $A^I{}_{\underline{u}}$.

(i.iv) $2\bar{n}+1$ functions $K^I, L_I, N$. They determine the functions $f, K_I$ and $H$, and, together with $\omega, \hat{A}^I$ and $A^I{}_{\underline{u}}$, the 1-forms $\psi$ via eq. (3.93) and the spatial 1-form $\omega$ via eq. (3.92) which can be written in the form

$$\hat{\star} F = \sqrt{3}(K^I \hat{\mathfrak{D}} L_I - L_I \hat{\mathfrak{D}} K^I) - \sqrt{3} K_I \partial_{\underline{u}} \hat{A}^I \,. \tag{3.103}$$

(ii) These objects must satisfy the following equations:

(ii.i) Eq. (3.62) that characterizes the quaternionic mappings $q^X$ and relates them to the spatial components of the gauge connection $\hat{A}^I$ and the functions $K^I$:

$$\hat{\mathfrak{D}}_r q^X J^r{}_X{}^Y = \sqrt{3} g K^I k_I{}^Y \,. \tag{3.104}$$

(ii.ii) Eq. (3.61) which relates the spatial components of the pullback of the SU(2) connection with the 3-dimensional spin connection, the spatial components of the gauge connection $\hat{A}^I$ and the functions $K^I$:

$$\varpi^{rs} = 2\varepsilon^{rst} \hat{B}^t - 2\sqrt{3} g K^I P_I^{[r} v^{s]} \,. \tag{3.105}$$

(ii.iii) Eq. (3.75) which relates the null component of the pullback of the SU(2) connection with the Dreibeins and the null components of the gauge connection $A^I{}_{\underline{u}}$:

$$v_{[r}{}^{\underline{r}} \partial_{\underline{u}} v_{s]\underline{r}} = -2\varepsilon_{rst} B^t{}_{\underline{u}} \,. \tag{3.106}$$

(ii.iv) Eq. (3.91), which follows from the Bianchi identity and can be put in the form

$$\hat{\mathfrak{D}} \hat{\star} \hat{\mathfrak{D}} K^I + \frac{2}{\sqrt{3}} g \hat{\mathfrak{D}} (\hat{\star} f^{-2} \hat{P}^I) = 0 \,. \tag{3.107}$$



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(ii.v) Eq. (3.100), the only independent Maxwell equation

$$
\begin{aligned}
&\hat{\mathfrak{D}}\hat{\star}\hat{\mathfrak{D}}L_I - g\hat{P}_I \wedge F + \frac{2}{\sqrt{3}} g C_{IJK}\hat{\mathfrak{D}}\hat{\star}(f^{-2}A^J{}_{\underline{u}}\hat{P}^K) \\
&\quad -gC_{IJK}\left[\hat{\mathfrak{D}}\hat{\star}(K^J\partial_{\underline{u}}\hat{A}^K) + \left(\hat{\mathfrak{D}}K^J + \frac{2}{\sqrt{3}}g\hat{\star}\hat{P}^J\right)\wedge\hat{\star}\partial_{\underline{u}}\hat{A}^K\right] \\
&\quad +\frac{1}{2}gf^{-3}\hat{\star}\left(k_{I\,x}\mathfrak{D}_{\underline{u}}\phi^x + k_{I\,X}\mathfrak{D}_{\underline{u}}q^X\right) = 0\,.
\end{aligned} \tag{3.108}
$$

(ii.vi) Eq. (3.102), the only independent component of the Einstein equations:

$$
\begin{aligned}
&\hat{d}\hat{\star}\hat{d}\,N - \frac{1}{\sqrt{3}}g\hat{\mathfrak{D}}\hat{\star}(f^{-2}C_{IJK}\hat{P}^I A^J_{\underline{u}}A^K{}_{\underline{u}}) \\
&\quad +\frac{1}{2}g\hat{\star}f^{-3}A^I{}_{\underline{u}}A^J{}_{\underline{u}}(g_{xy}k_I{}^x k_J{}^y + g_{XY}k_I{}^X k_J{}^Y) = 0\,.
\end{aligned} \tag{3.109}
$$

## 4. Conclusions

We have succeeded in finding a set of conditions which are necessary and sufficient for a configuration of gauged $N = 1, d = 5$ supergravity coupled to vector multiplets and hypermultiplets to be, first, supersymmetric and, second, a supersymmetric classical solution. As announced in the Introduction, the equations that we have obtained are highly non-linear and coupled, which does not seem to allow a systematic construction of non-trivial supersymmetric solutions. We leave the construction and study of examples for a future publication [23].

On the other hand, there exists an alternative supermultiplet for minimal supergravity in $d = 5$ [37−39]. it would be interesting to study the relations between the supersymmetric configurations we have found and those of the alternative formulation.

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## A. The gauging of isometries of the scalar manifolds

In this appendix we are going to review briefly the gauging of the isometries of the scalar manifolds of $N = 1, d = 5$ supergravity in order to clarify some definitions and conventions. This material is covered in a slightly different for in refs. [29] and [30].

### A.1 Killing vectors and gauge transformations

The complete scalar manifold (or target space) of the scalar fields of $N = 1, d = 5$ supergravity is the product of a real special manifold and a quaternionic Kähler manifold parametrized, respectively, by the scalars of the vector supermultiplets ($\phi^x$) and by the scalars of the hypermultiplets ($q^X$). The metrics of these two manifolds are denoted by $g_{xy}(\phi)$ and $g_{XY}(q)$.

We can describe the most general $N = 1, d = 5$ gauged supergravity theory by focusing on the gauging of the isometries of the scalar manifolds. In the end we will see that there are gaugings (necessarily Abelian) unrelated to isometries that fit in the general description.

The isometries to be gauged are generated by Killing vectors of the real special manifold $k_I{}^x(\phi)\partial_x$ and the quaternionic Kähler manifold $k_I{}^X(q)\partial_X$, a pair for each vector $A^I{}_\mu$ of the theory, although some (or all) can be identically zero.

The isometries generated by the Killing vectors $k_I{}^X$ act on the quaternions according to

$$\delta_\Lambda q^X = -g\Lambda^I k_I{}^X\,. \tag{A.1}$$

In the gauged theory the $\Lambda^I$s are the local parameters of vector gauge transformations

$$\delta_\Lambda A^I{}_\mu = \partial_\mu \Lambda^I + g f_{JK}{}^I A^J{}_\mu \Lambda^K\,, \tag{A.2}$$

where $f_{JK}{}^I$ are the structure constants of the gauge group $G$ and are given by the Lie brackets of the $k_I{}^X$s

$$[k_I, k_J] = -f_{IJ}{}^K k_K\,. \tag{A.3}$$

This implies that the functions $h^I$ of the real special manifold transform in the adjoint representation of $G$:

$$\delta_\Lambda h^I = -g f_{JK}{}^I \Lambda^J h^K\,. \tag{A.4}$$

In turn, this implies for the scalars themselves

$$\delta_\Lambda \phi^x = -g\Lambda^I k_I{}^x\,, \tag{A.5}$$

where

$$k_I{}^x = -\sqrt{3} f_{IJ}{}^K h^J h^x_K\,. \tag{A.6}$$

These objects must be Killing vectors of $g_{xy}(\phi)$ if the $\Lambda^I$ transformations are also symmetries of the corresponding $\sigma$ model. Writing $g_{xy}\partial\phi^x\partial\phi^y = -2C_{IJKL}h^I\partial h^J\partial h^K$ it is easy to see that necessary and sufficient condition is

$$f_{I(J}{}^K C_{MN)K} = 0\,, \tag{A.7}$$

i.e. that $C_{IJK}$ is an invariant tensor.

Furthermore, the Killing vectors $k_I{}^x(\phi)$ satisfy the same Lie algebra as the $k_I{}^X(q)$s and, using eq. (A.7), which implies

$$f_{IJ}{}^K h^J h_K = 0\,, \tag{A.8}$$

they can also be written in the equivalent form

$$k_I{}^x = -\sqrt{3} f_{IJ}{}^K h^{Jx} h_K \,. \tag{A.9}$$

The $G$-covariant derivatives on the scalars are

$$\begin{aligned} \mathfrak{D}_\mu \phi^x &= \partial_\mu \phi^x + g A^I{}_\mu k_I{}^x \,, \\ \mathfrak{D}_\mu h^I &= \partial_\mu h^I + g f_{JK}{}^I A^J{}_\mu h^K \,, \\ \mathfrak{D}_\mu q^X &= \partial_\mu q^X + g A^I{}_\mu k_I{}^X \,, \end{aligned} \tag{A.10}$$

and they transform covariantly as

$$\delta_\Lambda \mathfrak{D}_\mu \varphi^{\tilde{x}} = -g \Lambda^I \partial_{\tilde{y}} k_I{}^{\tilde{x}} \mathfrak{D}_\mu \varphi^{\tilde{y}} \,, \qquad \delta_\Lambda \mathfrak{D}_\mu h^I = -g f_{JK}{}^I \Lambda^J \mathfrak{D}_\mu h^K \,, \tag{A.11}$$

where we have unified the notation on the scalars, $\varphi^{\tilde{x}} = (\phi^x, q^X)$, $k_I{}^{\tilde{x}} = (k_I{}^x, k_I{}^X)$.

For the sake of completeness we also quote the formulae

$$\mathfrak{D}_\mu h_I = \partial_\mu h_I + g f_{IJ}{}^K A^J{}_\mu h_K \,, \qquad \mathfrak{D}_\mu C_{IJK} = 0 \,. \tag{A.12}$$

The second derivatives are defined by

$$\mathfrak{D}_\mu \mathfrak{D}_\nu \varphi^{\tilde{x}} \equiv \nabla_\mu \mathfrak{D}_\nu \varphi^{\tilde{x}} + \Gamma_{\tilde{y}\tilde{z}}{}^{\tilde{x}} \mathfrak{D}_\mu \varphi^{\tilde{y}} \mathfrak{D}_\mu \varphi^{\tilde{z}} + g A^I{}_\mu \partial_{\tilde{y}} k_I{}^{\tilde{x}} \mathfrak{D}_\nu \varphi^{\tilde{y}} \,, \tag{A.13}$$

where $\Gamma_{\tilde{y}\tilde{z}}{}^{\tilde{x}}$ are the target space Christoffel symbols. Their transformations and commutator are given by

$$\delta_\Lambda \mathfrak{D}_\mu \mathfrak{D}_\nu \varphi^{\tilde{x}} = -g \Lambda^I \partial_{\tilde{y}} k_I{}^{\tilde{x}} \mathfrak{D}_\mu \mathfrak{D}_\nu \varphi^{\tilde{y}} \,, \tag{A.14}$$

$$[\mathfrak{D}_\mu, \mathfrak{D}_\nu] \, \varphi^{\tilde{x}} = g F^I{}_{\mu\nu} k_I{}^{\tilde{x}} \,, \tag{A.15}$$

where $F^I{}_{\mu\nu}$ is the gauge field strength

$$F^I{}_{\mu\nu} = 2\partial_{[\mu} A^I{}_{\nu]} + g f_{JK}{}^I A^J{}_\mu A^K{}_\nu \,. \tag{A.16}$$

All these definitions are enough to construct a gauge-invariant action for the scalars, since this essentially depends on the target space metric. However, they are not enough to gauge the full supergravity theory, which depends on other structures as well. In particular, it depends on the complex structures of the hyperscalar manifold and we have to study under which conditions they are preserved by the gauging.

### A.2 The covariant Lie derivative and the momentum map

This appendix concerns only to the hyperscalar sector of the target manifold. The quaternionic Kähler geometry of this manifold is defined not only by the metric $g_{XY}$ but by the quaternionic structure $\vec{J}_X{}^Y$, which should also be preserved by the symmetries to be gauged. Therefore, one must require the vanishing of the Lie derivative of the quaternionic structure with respect to the Killing vectors $k_I{}^X$. One has to use an SU(2)-*covariant* Lie derivative for consistency or, as it is usually done in the literature, impose the vanishing of the standard Lie derivative up to gauge transformations. Here we will use an SU(2)-covariant Lie derivative whose construction we describe first.

Let $\vec{\psi}$ by an SU(2) vector and, simultaneously an arbitrary tensor on the hyperscalar variety, and $\vec{\omega}$ the SU(2) connection. Under infinitesimal SU(2) gauge transformations

$$\delta_\lambda \vec{\psi} = -2\vec{\lambda}(q) \times \vec{\psi}\,, \qquad \delta_\lambda \vec{\omega} = -2\vec{\lambda}(q) \times \vec{\omega} + d\vec{\lambda}(q)\,. \tag{A.17}$$

The standard Lie derivative of $\vec{\psi}$ along the vector $k_I{}^X$ (denoted by $\mathcal{L}_I\vec{\psi}$) transforms under SU(2) as

$$\delta_\lambda \mathcal{L}_I \vec{\psi} = -2\vec{\lambda} \times \mathcal{L}_I \vec{\psi} - 2\partial_I \vec{\lambda} \times \vec{\psi}\,, \tag{A.18}$$

where $\partial_I \equiv k_I{}^X \partial_X$. We now want to find another definition of Lie derivative that transforms without derivatives of the transformation parameter. Introducing for each Killing vector[11] $k_I{}^X$ a $\vec{\eta}_I$ transforming as

$$\delta_\lambda \vec{\eta}_I = -2\vec{\lambda} \times \vec{\eta}_I + \partial_I \vec{\lambda}\,, \tag{A.19}$$

we define the SU(2)-covariant Lie derivative on SU(2) vectors

$$\mathbb{L}_I \vec{\psi} \equiv \mathcal{L}_I \vec{\psi} + 2\vec{\eta}_I \times \vec{\psi}\,. \tag{A.20}$$

For this to be a good definition $\mathbb{L}_I$ must satisfy the standard properties of a Lie derivative.

$\mathbb{L}_I$ is clearly a linear operator and it satisfies the Leibnitz rule for products of SU(2) vectors such as $\vec{\psi} \cdot \vec{\phi}$ and $\vec{\psi} \times \vec{\phi}$. The Lie derivative must also satisfy

$$[\mathbb{L}_I, \mathbb{L}_J] = \mathbb{L}_{[k_I, k_J]}\,, \tag{A.21}$$

which implies the Jacobi identity. This requires the "curvature" of the "connection" $\vec{\eta}_I$ to be

$$\partial_I \vec{\eta}_J - \partial_J \vec{\eta}_I + 2\vec{\eta}_I \times \vec{\eta}_J = -f_{IJ}{}^K \vec{\eta}_K\,. \tag{A.22}$$

It should be clear that $\vec{\eta}_I$ must be related with the SU(2) connection $\vec{\omega}$, but it is not just $k_I{}^X \vec{\omega}_X$, which has the right transformation property eq. (A.19) but does not satisfy curvature property eq. (A.22). Thus, we introduce yet another SU(2) vector[12]

$$\vec{\eta}_I = k_I{}^X \vec{\omega}_X - \frac{1}{2}\vec{P}_I\,, \tag{A.23}$$

which must satisfy

$$\mathfrak{D}_I \vec{P}_J - \mathfrak{D}_J \vec{P}_I - \vec{P}_I \times \vec{P}_J + \frac{1}{2} k_I{}^X \vec{J}_{XY} k_J{}^Y = f_{IJ}{}^K \vec{P}_K\,, \tag{A.24}$$

in order to meet eq. (A.22). Here we have used the fact that in quaternionic Kähler manifolds the curvature of the SU(2) connection is non-vanishing and proportional to the

[11] Only covariant Lie derivatives with respect to Killing vectors can be properly defined.

[12] We put the $-1/2$ factor to agree with the conventions of ref. [30]

Kähler two-forms. We are going to show that $\vec{P}_I$ satisfies the equation that defines it as a *momentum map*.

Now, assuming that a $\vec{P}_I$ satisfying eq. (A.24) has been found, we can write the conditions that the vector $k_I{}^X$ must satisfy to be the generator of a symmetry of the hyperscalar manifold in the form

$$\mathbb{L}_I g_{XY} = 0\,, \tag{A.25}$$

$$\mathbb{L}_I \vec{J}_{XY} = 0\,. \tag{A.26}$$

The first equation is just the Killing equation since $\mathbb{L}_I g_{XY} = \mathcal{L}_I g_{XY}$. Given the metric and quaternionic structure, the second condition (*tri-holomorphicity* of the Killing vectors) can be seen as a condition for $\vec{P}_I$ just as the Killing equation can be seen as a condition for $k_I$ once the metric $g_{XY}$ is given: it can be written in the form

$$-\vec{J}_X{}^Y \times \vec{P}_I = \nabla_X k_I{}^Z \vec{J}_Z{}^Y - \vec{J}_X{}^Z \nabla_Z k_I{}^Y\,, \tag{A.27}$$

which says that $\vec{P}_I$ measures the commutator between the quaternionic structure and the covariant derivative of the Killing vectors. By contracting this equation with $\vec{J}_Y{}^X$ we obtain an expression for $\vec{P}_I$ itself, valid for $n_H \neq 0$[13]

$$2n_H \vec{P}_I = \vec{J}_X{}^Y \nabla_Y k_I{}^X\,. \tag{A.31}$$

For this solution to be consistent, it has to satisfy eq. (A.24). To see it we first take the derivative of the above solution eq. (A.31) using the following identity for Killing vectors,

$$\nabla_X \nabla_Y k^Z = R_{XWY}{}^Z k^W\,, \tag{A.32}$$

and the canonical decomposition of the curvature between its SU(2) and Sp($n_H$) parts,

$$R_{XWY}{}^Z = -\vec{J}_Y{}^Z \cdot \vec{\mathcal{R}}_{XW} + f_Y{}^{iB} f_{iA}{}^Z \mathcal{R}_{XW\,B}{}^A\,. \tag{A.33}$$

Only the SU(2) part of the curvature contributes to the derivative of $\vec{P}_I$:

$$\mathfrak{D}_X \vec{P}_I = 2\vec{\mathcal{R}}_{XY} k_I{}^Y = -\frac{1}{2}\vec{J}_{XY} k_I{}^Y\,. \tag{A.34}$$

[13]In absence of hypermultiplets ($n_H = 0$) the momentum map $\vec{P}_I$ can still be defined in two cases in which they are equivalent to a set of constant Fayet-Iliopoulos terms. In the first case the gauge group contains an SU(2) factor and

$$\vec{P}_I = \vec{e}_I\,\xi\,, \tag{A.28}$$

where $\xi$ is an arbitrary constant and the $\vec{e}_I$ are constants that are nonzero for $I$ in the range of the SU(2) factor and satisfy

$$\vec{e}_I \times \vec{e}_J = f_{IJ}{}^K \vec{e}_K\,. \tag{A.29}$$

In the second case the gauge group contains a U(1) factor and

$$\vec{P}_I = \vec{e}\,\xi_I\,, \tag{A.30}$$

where $\vec{e}$ is an arbitrary SU(2) vector and the $\xi_I$s are arbitrary constants that are nonzero for $I$ corresponding to the U(1) factor.



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This equation can alternatively be taken as the definition of $\vec{P}_I$. It defines a momentum map and it is crucial for coupling hypermultiplets to supergravity. Observe that the integrability condition of eq. (A.34) is precisely eq. (A.27).

We can now substitute eq. (A.34) in eq. (A.24), obtaining

$$\vec{P}_I \times \vec{P}_J + \frac{1}{2} k_I{}^X \vec{J}_{XY} k_J{}^Y = f_{IJ}{}^K \vec{P}_K \,. \tag{A.35}$$

On the other hand, contracting eq. (A.27) with $\nabla_Y k_J{}^X$ we get

$$n_H \vec{P}_I \times \vec{P}_J = -\vec{J}_X{}^Y \nabla_Y k_{[I|}{}^Z \nabla_Z k_{|J]}{}^X \,, \tag{A.36}$$

integrating by parts the right hand side of this expression, using the algebra of the Killing vectors, identity (A.32), the Bianchi identity of the curvature and the curvature decomposition (A.33) one recovers eq. (A.35).

From eq. (A.31) one can see that the momentum map is also covariantly preserved by the Killing vectors

$$\mathbb{L}_I \vec{P}_J = 0 \,. \tag{A.37}$$

There is still one more consistency check on the momentum map: the quaternionic Kähler two-form is SU(2)-covariantly closed. To ensure that this property is consistent with eq. (A.26) we must check that the covariant Lie derivative commutes with the SU(2)-covariant exterior derivative, in analogy to the commutation between standard Lie derivatives and exterior derivatives. This requirement leads us to the condition

$$\mathcal{L}_I \vec{\omega} - d\vec{\eta}_I - 2\vec{\omega} \times \vec{\eta}_I = 0 \,. \tag{A.38}$$

Notice that this relation between the two SU(2) connections is in principle independent of eq. (A.23). After substitution of eq. (A.23) in eq. (A.38) the latter becomes the differential definition of $\vec{P}_I$, eq. (A.34).

Eq. (A.34) can alternatively be used to solve the Killing vectors in terms of the derivatives of the momentum map,

$$k_I{}^X = \frac{2}{3} \vec{J}^{XY} \cdot \mathfrak{D}_Y \vec{P}_I \,. \tag{A.39}$$

In view of this relation $\vec{P}_I$ is sometimes called the prepotential.

The moment map assigns a triplet of real numbers to each Killing vector. The Killing vectors realize the algebra of $G$. Eq. (A.35) can also be understood as a realization of the algebra of $G$ in terms of $\vec{P}_I$, $\vec{J}_{XY}$ being the symplectic structure used to define the Poisson brackets which are the left hand side of eq. (A.35).

In summary, given a Killing vector of the metric $g_{XY}(q)$ we can always construct the momentum map $\vec{P}_I$ by eq. (A.31). Next we define the covariant Lie derivative along the Killing vector by means of the connection $\vec{\eta}_I$. This covariant Lie derivative enjoys the algebraic and differential properties of a pure Lie derivative and also commutes with covariant exterior derivatives. The Killing vector becomes automatically covariantly triholomorphic according to eq. (A.26).

### A.3 SU(2) transformations induced by $G$

Let us now consider the momentum map as a composite spacetime field over which depends only on the $q^X$s. Under general variations $\delta q^X$ and using the definition of the momentum map (A.34),

$$\delta \vec{P}_I = -\delta q^X \left( \frac{1}{2} \vec{J}_{XY} k_I{}^Y + 2\vec{\omega}_X \times \vec{P}_I \right) . \tag{A.40}$$

If this transformation is a $G$-gauge transformation $\delta_\Lambda q^X = -g\Lambda^J k_J{}^X$, taking into account eq. (A.35), we obtain

$$\delta_\Lambda \vec{P}_I = -g f_{IJ}{}^K \Lambda^J \vec{P}_K + 2g\Lambda^J \vec{\eta}_J \times \vec{P}_I \,, \tag{A.41}$$

which is the adjoint action of $G$ on $\vec{P}_I$ plus an induced SU(2) gauge transformation with parameter $-g\Lambda^J \vec{\eta}_J$ which is present even if $G$ is Abelian. This is the mechanism through which $G$ can act on objects such as the spinors of the supergravity theory which only have SU(2) indices, opening the doors to the gauging of groups larger than SU(2): if the gravitino transforms under standard SU(2) transformations according to

$$\delta_\lambda \psi^i_\mu = i\psi^j_\mu \vec{\sigma}_j{}^i \cdot \vec{\lambda} \,, \tag{A.42}$$

where $\vec{\lambda}$ is the infinitesimal SU(2) parameter, then, under $G$-gauge transformations it will undergo a similar transformation with $\vec{\lambda} = -g\Lambda^I \vec{\eta}_I$.

Thus, in $G$-gauged supergravity the pullback of the SU(2) connection that couples to the spinors of the theory has to be replaced by

$$\vec{B} \equiv \vec{A} + \frac{1}{2} g A^I \vec{P}_I \,, \qquad \vec{A} \equiv dq^X \vec{\omega}_X \,, \tag{A.43}$$

to take into account the SU(2) transformations induced by $G$-gauge transformations, which act on it as

$$\delta_\Lambda \vec{B} = -2(-g\Lambda^I \vec{\eta}) \times \vec{B} + d(-g\Lambda^I \vec{\eta}) \,. \tag{A.44}$$

The covariant derivative on these objects is

$$\mathfrak{D}_\mu \psi^i_\nu = \nabla_\mu \psi^i_\nu + \psi^j B_{\mu j}{}^i \,. \tag{A.45}$$

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